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THE 2-LENGTH OF THE HUGHES SUBGROUP

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ABSTRACT

For a finite group G and some prime power p^n, the H_{p^n}-subgroup $H_{p^n}(G)$ is defined by $H_{p^n}(G) = \langle x \in G \mid x^{p^n} \neq 1 \rangle$. Meixner proved that if G is a finite solvable group and $G \neq H_{2^n}(G)$ for some $n \geqq 1$, then the Fitting length of $H_{2^n}(G)$ is bounded by $4n$. In the following note it is shown that the 2-length of $H_{2^n}(G)$ is at most n. This result cannot be derived from Meixner's paper, since his result implies only that the 2-length is bounded by $2n$.

Let G be a finite solvable group and p a prime. Define the Hughes subgroup $H_{p^n}(G)$ of G by

$$H_{p^n}(G) = \langle x \in G : x^{p^n} \neq 1 \rangle.$$

In [5] Hartley and Rae proved that the Fitting length of $H_{p^n}(G)$ is bounded by some quadratic function of n if $H_{p^n}(G) \neq G$. Later this bound was improved by Kurzweil and Meixner ([8], [9]). For p odd Espuelas proved that the Fitting length of $H_{p^n}(G)$ is bounded by n and there are examples that prove that this is the best posible result ([2]). The purpose of this note is to prove that the 2-length of $H_{p^n}(G)$ does not exceed n. This result cannot be derived from Meixner's paper, since his result implies only that the 2-length is bounded by $2n$. We will prove the following Theorem.

THEOREM. *Let G be a finite solvable group having a normal subgroup $H \neq G$ such that every element of $G \setminus H$ is a 2-element of order at most 2^n. Then the 2-length $l_2(H)$ of H is at most n.*

In [1] Bryukhanova proved that the 2-length of a finite solvable group with Sylow-2-subgroups of exponent 2^n does not exceed n. Although the exponent of a Sylow-2-subgroup of H is generally not bounded (for example let G be a

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Dihedral group of order 2^{m+1}, then $H_{2^n}(G)$ is cyclic of order 2^m if $m > n$) we will see that the proof of the Theorem is similar to that in [1]. The notation of this note is standard and can be found in [6], [7].

PROOF OF THE THEOREM. Assume that the Theorem is false and let G be a minimal counterexample. Then we have the following:

(a) If U is a proper subgroup of G not contained in H then $l_2(U \cap H) \leqq n$ and hence $l_2(U) \leqq n + 1$.

(b) If U is a nontrivial normal subgroup of G contained in H then $l_2(H/U) \leqq n$.

(c) We have $n \geqq 2$, since for $n = 1$ the group H is nilpotent.

(d) Since $O_{2'}(G) \leqq H$ we have $O_{2'}(G) = 1$. Let $V = O_2(G)$ and $\bar{G} = G/V$. Then $V \leqq H$ and V is elementary abelian. Furthermore V is a faithful $\bar{G}$-module over GF(2).

(e) If $x \in G \setminus H$ and $y \in H$ then $(xy)^{2^n} = 1$. Thus if $x \in G \setminus H$ then the degree of the minimal polynomial of $\bar{x}$ on V is less than 2^n.

We need the following Lemma which is contained in [4].

LEMMA 1 (Hartley [4], Lemma 8). *Let G be a finite p-soluble group with upper p-series*

$$1 = G_0 \leqq O_{p'}(G) = G_1 \leqq O_{p'p}(G) = G_2 \leqq \cdots .$$

Let U be a subgroup of G having a series

$$1 = U_0 \leqq U_1 \leqq \cdots \leqq U_r = U \qquad (r \geqq 1)$$

such that U_i/U_{i-1} is a p'-group if i is odd and a p-group if i is even. Let $S_r \in \mathrm{Hall}_{\pi(U_r/U_{r-1})}(U)$ and for $r > i \geqq 1$ let $S_i \in \mathrm{Hall}_{\pi(U_i/U_{i-1})}([S_{i+1}, U_i]U_{i-1})$. If $U \leqq U_{r-1}G_{r-1}$, then $S_1 = 1$.

Let $1 = N_0 < N_1 < \cdots < N_{2n+1} = H$ be the upper 2-series of H and $G_k = G/N_{k-1}$, $H_k = H/N_{k-1}$ for $k = 1, \ldots, 2n + 1$.

LEMMA 2. *For $k = 1, \ldots, 2n + 1$ the groups H_k contain a nontrivial p_k-subgroup P_k (p_k a prime) which is normal in G_k such that:*

(a) *$p_{k+1} \neq p_k$, $p_k = 2$ for odd k and P_k is a special group.*

(b) *If R_{k+1} is any p_{k+1}-subgroup of N_{k+1} which covers P_{k+1} then*

  (i) *$P_k = [P_k, R_{k+1}]$.*

  (ii) *R_{k+1} centralizes each proper subgroup of P_k that is normal in G_k.*

  (iii) *$N_k/N_{k-1} = \mathbf{C}_{N_k/N_{k-1}}(R_{k+1})P_k$.*

Proof. We will define the groups P_k inductively. Let $P_{2n+1} = H_{2n+1}$ and assume that we have already defined $P_{2n+1}, \dots, P_l$. Let R_l be a p_l-subgroup of N_l such that $R_lN_{l-1}/N_{l-1} = P_l$. Since

$$\mathbf{C}_H(N_{l-1}/N_{l-2}) \leqq N_{l-1}$$

we have $[R_1, N_{l-1}] \nleqq N_{l-2}$. Since the orders of R_l and N_{l-1}/N_{l-2} are coprime, there exists a Sylow-p-subgroup P of N_{l-1}/N_{l-2} which is normalized but not centralized by R_l. Let X denote the preimage of $\mathbf{N}_{G_{l-1}}(P)$ in G. Then $G = N_{l-1}X$ by the Frattini argument.

Let $1 = X_0 < X_1 < X_2 < \cdots$ be the upper 2-series of $X \cap H$. Since $N_1 \leqq X$ we have $O_{2'}(X \cap H) = 1$. Note that

$$R_i \leqq N_i \leqq X_iN_{l-1} \qquad \text{for } i = l, \dots, 2n+1.$$

We will prove by induction that

$$(*) \qquad R_{i+1} \nleqq X_iN_{l-1} \qquad \text{for } i = l, \dots, 2n+1.$$

Assume that $R_l \leqq X_{l-1}N_{l-1}$. Then by comparison of the orders of the groups R_l, X_{l-1}/X_{l-2} and N_{l-1}/N_{l-2} we have

$$R_l \leqq X_{l-2}.$$

Hence $[P, R_l] = 1$ by Lemma 1. Therefore $R_l \nleqq X_{l-1}N_{l-1}$. Now $(*)$ is proved by induction. Thus R_{2n+1} is not contained in $X_{2n}N_{l-1}$ and hence $l_2(X) = n + 1$. Since $N_{l-1} \leqq H \neq G$ we have $X \nleqq H$. Therefore by the choice of the counterexample $G = X$ and $P \trianglelefteq G_{l-1}$. Let P_{l-1} be a minimal normal subgroup of G_{l-1} contained in P such that $[P_{l-1}, R_l] \neq 1$.

Let $\hat{R}_l = R_lN_{l-2}/N_{l-2}$. Then $\hat{R}_l$ is a Sylow-p_l-subgroup of the normal subgroup R_lN_{l-1}/N_{l-2} of G_{l-1}. Hence by the Frattini argument:

$$G_{l-1} = \mathbf{N}_{G_{l-1}}(\hat{R}_l)N_{l-1}/N_{l-2}.$$

With the same argument as above we prove that the 2-length of the preimage of $P_{l-1}\mathbf{N}_{G_{l-1}}(\hat{R}_l)$ in G is $n + 1$. Hence by the choice of the counterexample

$$G_{l-1} = P_{l-1}\mathbf{N}_{G_{l-1}}(\hat{R}_l).$$

Now we are in the position to prove that P_{l-1} has all the desired properties for the particular choice of R_l. Since $[P_{l-1}, R_l] \trianglelefteq P_{l-1} \trianglelefteq G_{l-1}$ we have by minimality of P_l

$$P_{l-1} = [P_{l-1}, R_l]$$

and by [6], III.13.6, we know that P_{l-1} is special. Since $N_{l-1}/N_{l-2} = P_{l-1}\mathbf{N}_{N_{l-1}/N_{l-2}}(\hat{R}_l)$ and

$$[\mathbf{N}_{N_{l-1}/N_{l-2}}(\hat{R}_l), \hat{R}_l] \leqq N_l/N_{l-2} \cap \hat{R}_l = 1,$$

we have

$$\mathbf{N}_{N_{l-1}/N_{l-2}}(\hat{R}_l) = \mathbf{C}_{N_{l-1}/N_{l-2}}(\hat{R}_l).$$

Now let S be any p_l-subgroup of G such that $SN_{l-1}/N_{l-1} = P_l$. Since orders of R_l and N_{l-1}/N_{l-2} are coprime, SN_{l-2} and R_lN_{l-2} are conjugate by an element g. Therefore

$$[P_{l-1}, S] = [P_{l-1}, R_l]^g = P_{l-1}^g = P_{l-1}.$$

Thus the Lemma is proved.

Let us denote P_2 by Q. Then Q is a special q-group of exponent q. Let S be a Sylow-2-subgroup of $O_{2'2}(H/V)$ and $T \leqq S$ such that $TO_{2'}(\bar{H})/O_{2'}(\bar{H}) = P_3$. Since $O_{2'22'}(\bar{H}) = O_{2'}(\bar{H})\mathbf{N}(S)$, there exists a p_4-subgroup P of $\mathbf{N}(S)$ such that

$$P_4 = PO_{2'2}(\bar{H})/O_{2'2}(\bar{H}).$$

LEMMA 3. (a) *Q is nonabelian and there exists a natural number m such that $q = 2^m - 1$.*

(b) *$\bar{G}$ acts irreducibly on V and $C_V(Q) = 1$.*

(c) *$V \rtimes \bar{G}$ is a counterexample.*

PROOF. (a) If $x \in \bar{G} \setminus \bar{H}$ then $C_{\bar{G}}(x)$ is a 2-group and hence $C_{\bar{G}}(Q) = C_{\bar{H}}(Q)$. Let $X = Q \rtimes \bar{G}/C_{\bar{G}}(Q)$. Then by Lemma 2, $l_2(X) = n$. By the choice of the counterexample, there exists an element x in $\bar{G} \setminus \bar{H}$ of order 2^n such that

$$[x^{2^{n-1}}, Q] \neq 1.$$

Since the degree of the minimal polynomial of x on V is less than 2^n we can apply the Theorem of Hall and Higman ([7], IX, 9.2).

(b) Let N be a normal subgroup of G contained in V such that $[V, Q] \nleqq N$. By Lemma 2 we have $l_2(H/N) = n + 1$. Hence by the choice of the counterexample $N = 1$. Therefore we have $C_V(Q) = 1$. Let $V_0 \neq V$ be a $\bar{G}$-submodule of V. Since $C_V(Q) = 1$ we have $C_Q(V/V_0) \neq Q$. This implies $V_0 = 1$.

(c) Since $VR_2 \trianglelefteq G$, we have $G = V\mathbf{N}_G(R_2)$. Because of $[\mathbf{N}_V(R_2), R_2] = 1$, we have

$$\mathbf{N}_V(R_2) = \mathbf{C}_V(R_2) \leqq \mathbf{C}_V(Q) = 1.$$

Thus G splits over V.

Let $B = \langle x^{2^{n-1}} \mid x \in \mathrm{N}_G(S)\setminus H\rangle$. If $x \in \mathrm{N}_G(S)\setminus H$ then $\langle T, x\rangle$ is a 2-group. Now $TO_{2'}(\bar G) \trianglelefteq \bar G$, thus $T \trianglelefteq \langle T, x\rangle$ and therefore $x \in N_G(T)$. This yields $B \leqq N_G(T)$.

Lemma 4. (a) *$T \leqq B \leqq S$ and $O_{2'}(\bar G)B \trianglelefteq \bar G$.*
(b) $l_2(H/O_{2'}(\bar G)B) = n - 1$.

Proof. By the Frattini argument we have $\bar G = \mathrm{N}_G(S)O_{2'}(\bar G)$. By [3], Theorem B, we have

$$B \leqq O_{2'2}(\bar G) \cap \mathrm{N}_G(S)$$

and hence $B \leqq S$ since $S \in \mathrm{Syl}_2(O_{2'2}(\bar G))$. Because B is a normal subgroup of $\mathrm{N}_G(S)$ we have $O_{2'}(\bar G)B \trianglelefteq \bar G$. Assume that $T \nleqq B$ and let $\hat H = \bar H/O_{2'}(\bar G)B$. By Lemma 2 we have

$$TO_{2'}(\bar G)/O_{2'}(\bar G) \cap BO_{2'}(\bar G)/O_{2'}(\bar G) \leqq \Phi(TO_{2'}(\bar G)/O_{2'}(\bar G))$$

and therefore using Lemma 2 again

$$l_2(\hat H) = n.$$

Obviously the order of every element of $\bar G/O_{2'}(\bar G)B\setminus \hat H$ divides 2^{n-1}. This contradicts the choice of the counterexample. Hence $T \leqq B$ and $l_2(\hat H) = n - 1$.

The following Lemma is now proved on the same lines as Lemma 23 in [1].

Lemma 5. *Let $g \in \mathrm{N}_G(T)\setminus \bar H$ and $x \in T$. If $u = g^{2^{n-1}}(gx)^{2^{n-1}}$ then one of the following holds.*
(a) $[u, Q] = 1$.
(b) *T is nonabelian and* $[[T, u], Q] = 1$.

Now we are in the position to prove the Theorem. Let

$$\hat G = \bar G/C_G(Q), \quad \hat T = TC_G(Q)/C_G(Q), \quad F = \mathrm{N}_{\hat G}(\hat T)/\mathrm{C}_{\hat H}(\hat T/\Phi(\hat T))$$

and

$$U = \hat T/\Phi(\hat T) \rtimes F.$$

Furthermore let

$$\hat H = \bar H/C_G(Q), \quad \tilde F = \mathrm{N}_{\hat H}(\hat T)/\mathrm{C}_{\hat H}(\hat T/\Phi(\hat T)) \quad \text{and} \quad U_1 = \hat T/\Phi(\hat T) \rtimes \tilde F.$$

We will prove that the order of every element of $U \setminus U_1$ divides 2^{n-1} and that $l_2(U_1) = n$. This contradicts the choice of the counterexample and completes the proof of the Theorem.

(a) $l_2(U_1) = n$. By the Frattini argument we have

$$\hat{H} = O_{2'}(\bar{G})C_G(Q)/C_G(Q)\mathrm{N}_{\hat{H}}(\hat{T})$$

thus

$$\hat{H} = O_{2'}(\hat{H})\mathrm{N}_{\hat{H}}(\hat{T})$$

and hence

$$l_2(N_{\hat{H}}(\hat{T})) = l_2(\hat{H}) = n.$$

Let $\hat{R}_i$ denote the image of R_i in $\tilde{F}$. Then we can assume that the $\hat{R}_i$ satisfy the same conditions as the R_i. Since $\hat{T}/\Phi(\hat{T}) \leqq O_2(U_1)$, we have $O_{2'}(U_1) = 1$.

Let $1 = F_0 \leqq F_1 = O_2(\tilde{F}) \leqq F_2 \cdots$ be the upper 2-series of $\tilde{\mathrm{F}}$. Analogously to Lemma 2 we will prove that

$$(**) \qquad \hat{R}_i \leqq F_{i-2} \quad \text{and} \quad \hat{R}_i \nleqq F_{i-3} \qquad \text{for } i \geqq 4.$$

Assume that $\hat{R}_4 \leqq F_1$, then $[R_4, T] \leqq \mathrm{C}_G(Q)$ since the order of $\hat{R}_4$ is odd. Thus

$$TO_{2'}(\bar{G}) = [R_4, T]O_{2'}(\bar{G}) \leqq \mathrm{C}_G(Q)O_{2'}(\bar{G}).$$

This implies $T \leqq \mathrm{C}_G(Q)$ since Q is a normal subgroup of $\bar{G}$. This contradiction proves $\hat{R}_4 \nleqq F_1$.

Now (**) is proved by induction using Lemma 2. Thus $\hat{R}_{2n-1} \nleqq F_{2(n-1)}$ and therefore $l_2(U_1) = n$.

(b) $[B, T] \leqq \Phi(T)$. Since $B \leqq \mathrm{N}_G(T)$ and $B \leqq S$ we have

$$[B, T] < T.$$

Thus $[T, B]O_{2'}(\bar{G})/O_{2'}(\bar{G})$ is a normal subgroup of H_3 properly contained in P_3. Hence by Lemma 2

$$[B, T, P] \leqq O_{2'}(\bar{G}).$$

Therefore $[B, T, P] = 1$ since $P \leqq \mathrm{N}(S)$. Now $TO_{2'}(\bar{G}) = [T, P]O_{2'}(\bar{G})$ implies $[T, P] = T$. This yields

$$1 = [[T, B]\Phi(T)/\Phi(T), P] = [T, B]\Phi(T)/\Phi(T).$$

(c) $\hat{T}$ *is special*. Since T is special we can assume that $\hat{T}$ is nonabelian. Let X be the preimage of $Z(T/C_T(Q))$. Then $X < T$ and $XO_{2'}(\bar{G}) \trianglelefteq \bar{G}$ since $G = O_{2'}(G)\mathrm{N}_G(T)$. Thus $[X, P] \leqq O_{2'}(\bar{G})$ by Lemma 2 and hence $[X, P] = 1$. Since $T = [T, P]$, we have $X \leqq \Phi(T) = Z(T)$. Thus $Z(\hat{T}) = \Phi(\hat{T})$ and hence $\hat{T}$ is special.

(d) *If $x \in U \setminus U_1$, then the order of x divides* 2^{n-1}. Let $W \in \mathrm{Syl}_2(\mathrm{N}_G(T))$ such that $S \leqq W$ and let $\bar{W}$ be the image of W in F. Then $\hat{T}/\Phi(\hat{T}) \rtimes \bar{W} \in \mathrm{Syl}_2(U)$. WLOG we can assume that x is contained in that group. If g is an element in $\mathrm{N}_G(T) \setminus \bar{H}$, then g is conjugate to an element in $\mathrm{N}_G(S) \setminus \bar{H}$. Therefore the order of the image of g in F is less than or equal to 2^{n-1}. Now let $x \in T$ and $g \in \mathrm{N}_G(T) \setminus \bar{H}$. We have

$$(gx)^{2^{n-1}} = g^{2^{n-1}}u,$$

where u is defined as in Lemma 5. To prove (d) it remains to prove that the image of u in T is contained in $\Phi(T)$. But this is a consequence of Lemma 5 and (c):

If $[u, Q] = 1$, then the image of u in $\hat{T}$ is 1 and if $[T, u, Q] = 1$, then the image of u is contained in $Z(\hat{T}) = \Phi(\hat{T})$.

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